Group foliation and non-invariant solutions of the heavenly equation

Group foliation and non-invariant solutions of the heavenly equation

L Martina[1], M B Sheftel[2] and P Winternitz[3]

[1] Dipartimento di Fisica dell'Università di Lecce and Sezione INFN-Lecce, CP 193,
73100 Lecce, Italy
[2] Feza Gürsey Institute, PO Box 6, Cengelkoy, 81220 Istanbul, Turkey and Department of Higher
Mathematics, North Western State Technical University, Millionnaya Street 5, 191186,
St Petersburg, Russia
[3] Centre de Recherches Mathématiques and Département de mathématiques et statistique,
Université de Montréal, CP 6128-CV, Montréal, QC, Canada H3C3J7

E-mail: Luigi.Martina@le.infn.it, sheftel@gursey.gov.tr and wintern@CRM.UMontreal.ca

Abstract
The main physical results of this paper are new exact analytical solutions of
the heavenly equation, of importance in the general theory of relativity. These
solutions are not invariant under any subgroup of the symmetry group of the
equation. The main mathematical result is a new method of obtaining non-
invariant solutions of partial differential equations with infinite-dimensional
symmetry groups. The method involves the compatibility of the given equations
with a differential constraint, which is automorphic under a specific symmetry
subgroup, the latter acting transitively on the submanifold of the common
solutions. By studying the integrability of the resulting conditions, one can
provide an explicit foliation of the entire solution manifold of the considered
equations.

PACS numbers: 02.20.Tw, 02.30.Jr, 04.20.Jb, 03.65.Fd

1. Introduction

An important problem for partial differential equations invariant with respect to an infinite Lie
group is to obtain non-invariant solutions that admit no continuous symmetries of the equations.
In our opinion, the old approach of Lie [1] developed by Vessiot [2] and in modern form by
Ovsiannikov [3], which we call *group foliation*, is an adequate tool for treating this problem
in the framework of Lie theory. According to this method we foliate the solution space of the
equations in question into orbits, choosing an infinite-dimensional symmetry group for the
foliation. Each orbit is determined by the *automorphic system* joined to the original equations
and considered as invariant differential constraints. Due to the automorphic property of this

system, any of its solutions can be obtained from any other solution by a transformation of the chosen symmetry subgroup. This symmetry property makes the automorphic system completely integrable if only one of its solutions can be obtained. The collection of orbits of all solutions of the original equations is determined by the *resolving system*. Thus the problem reduces to obtaining as many particular solutions of the resolving system as possible. Each of them will fix a particular automorphic system and the corresponding orbit in the solution space of original equations.

Group theory is usually used to obtain invariant solutions. Here we show that it also provides a mechanism for obtaining non-invariant solutions. We give examples of such solutions as an application of the method.

In this paper we further develop the method of group foliation by introducing a procedure of *invariant integration*. It is used for reconstructing the solution of the original equation corresponding to the known particular solution of the resolving system. We apply the method for obtaining non-invariant solutions of the 'heavenly' equation

$$u_{xx} + u_{yy} = \kappa (e^u)_{tt} \tag{1.1}$$

where $\kappa = \pm 1$ and the unknown u depends on the time t and two space variables x and y. Here and further subscripts of u denote partial differentiation with respect to corresponding variables. This equation formally is a continuous version of the Toda lattice [4]. It appears in various physical theories, such as the theory of area preserving diffeomorphisms [5], in the theory of the so-called gravitational instantons [6] and in the general theory of relativity [7]. In this context, it describes self-dual Einstein spaces with Euclidean signature with one rotational Killing vector [8]. Moreover, it is a completely integrable system in the sense of the existence of a Lax pair [9, 10].

The outline of the method is as follows. We determine the total group of point symmetries of the heavenly equation. For the group foliation we choose its infinite subgroup of conformal transformations. We compute differential invariants of this subgroup up to the second order inclusively and obtain five functionally independent differential invariants. On account of the heavenly equation we are left with four invariants. We choose three of them as new independent variables, the same number as in the heavenly equation, and one is left for the new unknown.

We obtain three first-order *operators of invariant differentiation* defined by the property that acting on a differential invariant they again produce a differential invariant. These operators are determined by the condition that they should commute with an arbitrary prolongation of any element of the infinite symmetry Lie algebra chosen for the foliation.

Extensive use of operators of invariant differentiation and their commutator algebra for formulating the resolving system is a new feature of the method suggested by one of the authors (MBS) in a recent paper on the complex Monge–Ampère equation [11]. Equation (1.1) is itself a symmetry reduction of the complex Monge–Ampère equation and is often called the Boyer–Finley equation. We derive the resolving system as a set of compatibility conditions for the heavenly equation and its automorphic system using invariant differentiation. Then we formulate the resolving system in terms of the *commutator algebra of operators of invariant differentiation* by discovering the fact that this algebra together with its Jacobi identities, projected on the solution manifold of the considered equation in the space of differential invariants, is equivalent to the resolving system.

We show how an ansatz simplifying the commutator algebra of operators of invariant differentiation leads to a particular class of solutions of the resolving system. Then we use invariant integration to obtain the corresponding solution of the heavenly equation and prove that this solution is non-invariant.

2. Lie group of point symmetries and differential invariants

It is convenient to work with the heavenly equation using the complex coordinates $z = (x + \mathrm{i}y)/2$ and $\bar{z} = (x - \mathrm{i}y)/2$,

$$u_{z\bar{z}} = \kappa(\mathrm{e}^u)_{tt}. \tag{2.1}$$

A standard calculation of the total symmetry group of the heavenly equation gives the following result for the symmetry generators of all one-parameter subgroups [12]:

$$
\begin{aligned}
T &= \partial_t \qquad G = t\partial_t + 2\partial_u \\
X_a &= a(z)\partial_z + \bar{a}(\bar{z})\partial_{\bar{z}} - (a'(z) + \bar{a}'(\bar{z}))\partial_u
\end{aligned}
\tag{2.2}
$$

where T is the generator of translations in t, G is the generator of a dilation of time accompanied by a shift of u: $t = \tilde{t}\mathrm{e}^\tau$, $u = \tilde{u} + 2\tau$ and X_a is a generator of the conformal transformations

$$z = \phi(\tilde{z}) \qquad \bar{z} = \bar{\phi}(\tilde{\bar{z}}) \qquad u(z, \bar{z}, t) = \tilde{u}(\tilde{z}, \tilde{\bar{z}}, t) - \ln(\phi'(\tilde{z})\bar{\phi}'(\tilde{\bar{z}})) \tag{2.3}$$

where $a(z)$ and $\phi(z)$ are arbitrary holomorphic functions of z (see also [13]).

The Lie algebra of the symmetry generators is determined by the commutation relations

$$[T, G] = T \qquad [T, X_a] = 0 \qquad [G, X_a] = 0 \qquad [X_a, X_b] = X_{ab'-ba'} \tag{2.4}$$

which show that the generators X_a of conformal transformations form an infinite-dimensional subalgebra.

We choose for the group foliation the corresponding infinite symmetry subgroup of all holomorphic transformations in z, i.e. the conformal group. Differential invariants of this group are the invariants of all its generators X_a of the form (2.2) in the prolongation spaces. This means that they can depend on independent variables, on the unknowns and also on the partial derivatives of the unknowns allowed by the order of the prolongation. The order N of the differential invariant is defined as the order of the highest derivative which this invariant depends on. The determining equation for differential invariants Φ of the order $N \leqslant 2$ has the form

$$\overset{2}{X}(\Phi) = 0 \tag{2.5}$$

where $\overset{2}{X}$ is the second prolongation of the generator X_a (2.2) of the conformal group defined by the standard prolongation formulae,

$$
\begin{aligned}
\overset{2}{X} = {}& a\partial_z + \bar{a}\partial_{\bar{z}} - (a' + \bar{a}')\partial_u - (a'' + a'u_z)\partial_{u_z} - (\bar{a}'' + \bar{a}'u_{\bar{z}})\partial_{u_{\bar{z}}} \\
& - (a''' + a''u_z + 2a'u_{zz})\partial_{u_{zz}} - (\bar{a}''' + \bar{a}''u_{\bar{z}} + 2\bar{a}'u_{\bar{z}\bar{z}})\partial_{u_{\bar{z}\bar{z}}} \\
& - a'u_{zt}\partial_{u_{zt}} - \bar{a}'u_{\bar{z}t}\partial_{u_{\bar{z}t}} - (a' + \bar{a}')u_{z\bar{z}}\partial_{u_{z\bar{z}}}
\end{aligned}
\tag{2.6}
$$

where $a = a(z)$ and $\bar{a} = \bar{a}(\bar{z})$.

The integration of equation (2.5) gives five functionally independent differential invariants up to second order inclusively,

$$t, \quad u_t, \quad u_{tt}, \quad \rho = \mathrm{e}^{-u}u_{z\bar{z}}, \quad \eta = \mathrm{e}^{-u}u_{zt}u_{\bar{z}t} \tag{2.7}$$

and all of them turn out to be real. This allows us to express the heavenly equation (2.1) solely in terms of the differential invariants

$$u_{tt} = \kappa\rho - u_t^2. \tag{2.8}$$

3. Operators of invariant differentiation and a basis of differential invariants

Operators of invariant differentiation are linear combinations of total derivative operators with respect to independent variables. Their coefficients depend on local coordinates of the prolongation space. They are defined by the special property that, acting on any (differential) invariant, they map it again into a differential invariant. Being first-order differential operators, they raise the order of a differential invariant by 1. Invariance requires that these differential operators commute with any infinitely prolonged generator X_a (2.2) of the conformal symmetry group. It is obvious (see [3] para. 24.2 for a complete proof) that the total number of independent operators of invariant differentiation is equal to the number of total derivative operators, that is to the number of independent variables (which is three in the present case).

We look for operators of invariant differentiation in the form

$$\delta = \lambda_1 D_t + \lambda_2 D_z + \lambda_3 D_{\bar{z}} = \sum_{i=1}^{3} \lambda_i D_i \tag{3.1}$$

where $D_1 = D_t$, $D_2 = D_z$, $D_3 = D_{\bar{z}}$ are operators of total derivatives with respect to the subscripts. We look for the coefficients λ_i satisfying the condition of commutativity of δ with the infinite prolongation $\overset{\infty}{X}$ of the generator X_a (2.2). It can be decomposed as the sum of the infinite prolongation of the symmetry generator in the evolution form $\hat{X}$ [14] and the linear combination of the total derivative operators,

$$\overset{\infty}{X} = \hat{X} + \sum_{j=1}^{3} \xi^j D_j = \hat{X} + a(z) D_z + \bar{a}(z) D_{\bar{z}} \tag{3.2}$$

where from the form of X_a we take

$$\xi^1 = \xi^t = 0 \qquad \xi^2 = \xi^z = a(z) \qquad \xi^3 = \xi^{\bar{z}} = \bar{a}(\bar{z}). \tag{3.3}$$

The generator $\hat{X}$ in the evolution form commutes with all the total derivatives D_j: $[D_i, \hat{X}] = 0$ and hence we have the standard commutation relation

$$\left[D_i, \overset{\infty}{X} \right] = \left[D_i, \sum_{j=1}^{3} \xi^j D_j \right] = \sum_{j=1}^{3} D_i(\xi^j) D_j. \tag{3.4}$$

We use it in the determining relation for operators of invariant differentiation

$$\left[\delta, \overset{\infty}{X} \right] = \sum_i \left[\lambda_i D_i, \overset{\infty}{X} \right] = \sum_{i,j} \left(\lambda_i D_i[\xi^j] D_j - \overset{\infty}{X}(\lambda_i) D_i \right)$$

$$= \sum_{i,j} \left(\lambda_j D_j[\xi^i] - \overset{\infty}{X}(\lambda_i) \right) D_i = 0. \tag{3.5}$$

The final equation for the coefficients λ_i of the operators of invariant differentiation (see equation (24.2.3) of [3]) is

$$\overset{\infty}{X}(\lambda_i) = \sum_{j=1}^{3} \lambda_j D_j[\xi^i]. \tag{3.6}$$

Using (3.3) and restricting ourselves to the second prolongation $\overset{2}{X}$ of the symmetry generator, equation (3.6) leads to

$$\overset{2}{X}(\lambda_1) = 0 \qquad \overset{2}{X}(\lambda_2) = \lambda_2 a'(z) \qquad \overset{2}{X}(\lambda_3) = \lambda_3 \bar{a}'(\bar{z}) \tag{3.7}$$

where primes denote derivatives and

$$\lambda_i = \lambda_i(t, z, \bar{z}, u, u_t, u_z, u_{\bar{z}}, u_{zt}, u_{\bar{z}t}, u_{z\bar{z}}, u_{zz}, u_{\bar{z}\bar{z}}).$$

Here $\overset{2}{X}$ is the second prolongation of the generator X_a of the conformal group defined by equation (2.6).

Equations (3.7) are easily solved by the method of characteristics and we choose three of the simplest, linearly independent solutions for the coefficients λ_i of the three operators of invariant differentiation

$$\begin{array}{llllll} \lambda_1^1 = 1 & \lambda_2^1 = 0 & \lambda_3^1 = 0 & \lambda_1^2 = 0 & \lambda_2^2 = e^{-u} u_{\bar{z}t} & \lambda_3^2 = 0 \\ \lambda_1^3 = 0 & \lambda_2^3 = 0 & \lambda_3^3 = e^{-u} u_{zt}. \end{array} \tag{3.8}$$

From here we obtain a basis for the operators of invariant differentiation

$$\delta = D_t \qquad \Delta = e^{-u} u_{\bar{z}t} D_z \qquad \bar{\Delta} = e^{-u} u_{zt} D_{\bar{z}}. \tag{3.9}$$

The *basis of differential invariants* is defined as a minimal finite set of invariants of a symmetry group from which any other differential invariant of this group can be obtained by a finite number of invariant differentiations and operations of taking composite functions. The proof of the existence and finiteness of the basis was given by Tresse [15] and in a more modern form by Ovsiannikov [3].

In our example, the basis of differential invariants is formed by the set of three invariants t, u_t, ρ, while two other invariants u_{tt} and η of equation (2.7) are given by the relations

$$u_{tt} = \delta(u_t) \qquad \eta \equiv e^{-u} u_{zt} u_{\bar{z}t} = \Delta(u_t) = \bar{\Delta}(u_t). \tag{3.10}$$

All other functionally independent higher order invariants can be obtained by acting with operators of invariant differentiation on the basis $\{t, u_t, \rho\}$. In particular, the following third-order invariants generated from the second-order invariant ρ by invariant differentiations will be involved in our construction:

$$\sigma = \Delta(\rho) \qquad \bar{\sigma} = \bar{\Delta}(\rho) \qquad \tau = \delta(\rho) \equiv \rho_t. \tag{3.11}$$

The operators of invariant differentiation form the commutator algebra

$$[\delta, \Delta] = \left(\kappa \frac{\bar{\sigma}}{\eta} - 3u_t\right)\Delta \qquad [\delta, \bar{\Delta}] = \left(\kappa \frac{\sigma}{\eta} - 3u_t\right)\bar{\Delta}$$
$$[\Delta, \bar{\Delta}] = \left(\frac{\Delta(\eta)}{\eta} - (u_t\rho + \tau)\right)\bar{\Delta} - \left(\frac{\bar{\Delta}(\eta)}{\eta} - (u_t\rho + \tau)\right)\Delta \tag{3.12}$$

which form a Lie algebra over the field of invariants of the conformal group, in agreement with Ovsiannikov's lemma 24.2 [3].

The commutator algebra is simplified by introducing two new operators of invariant differentiation Y and $\bar{Y}$ instead of Δ and $\bar{\Delta}$ and two new variables λ and $\bar{\lambda}$ instead of σ and $\bar{\sigma}$, defined by

$$\Delta = \eta Y \qquad \bar{\Delta} = \eta \bar{Y} \qquad \sigma = \eta \lambda \qquad \bar{\sigma} = \eta \bar{\lambda} \tag{3.13}$$

and becomes

$$[\delta, Y] = \left(\kappa \bar{\lambda} - 3u_t - \frac{\delta(\eta)}{\eta}\right)Y \qquad [\delta, \bar{Y}] = \left(\kappa \lambda - 3u_t - \frac{\delta(\eta)}{\eta}\right)\bar{Y}$$
$$[Y, \bar{Y}] = \frac{(u_t\rho + \tau)}{\eta}(Y - \bar{Y}). \tag{3.14}$$

Equations (3.10) and (3.11) imply the following properties of the operators Y and $\bar{Y}$:

$$Y(u_t) = \bar{Y}(u_t) = 1 \qquad Y(\rho) = \lambda \qquad \bar{Y}(\rho) = \bar{\lambda}. \tag{3.15}$$

4. Automorphic and resolving equations

We have four independent differential invariants t, u_t, ρ, η on the solution manifold of the heavenly equation (2.8). We choose three of them, t, u_t, ρ, as new invariant independent variables, the same number as in the original equation (2.1), and consider the fourth one η as a real function F of these three

$$\eta = F(t, u_t, \rho) \iff u_{zt}u_{\bar{z}t}\mathrm{e}^{-u} = F(t, u_t, u_{z\bar{z}}\mathrm{e}^{-u}) \tag{4.1}$$

which gives us the general form of the automorphic equation, i.e. invariant differential constraint.

Our next task is to derive the *resolving equations* for the heavenly equation. This will account for all integrability conditions of the system (2.8), (4.1) in an explicitly invariant form. If we pick a particular solution of this *resolving system* for F and use it on the right-hand side of (4.1), then the latter equation will possess the automorphic property: each solution of it can be obtained from any other solution by an appropriate conformal symmetry transformation.

We consider the automorphic equation (4.1) divided by F in the form

$$Y(u_t) = 1 \tag{4.2}$$

and the heavenly equation (2.8) in the form

$$\delta(u_t) = \kappa\rho - u_t^2. \tag{4.3}$$

We put $\eta = F$ in the definitions (3.13) of Y and $\bar{Y}$ and in their commutation relations (3.14). The integrability condition for the system (4.2) and (4.3) is obtained by the *invariant cross-differentiation* with δ and Y with the use of their commutation relation (3.14),

$$\delta(F) = [\kappa(\lambda + \bar{\lambda}) - 5u_t]F. \tag{4.4}$$

Since this equation involves λ and $\bar{\lambda}$ we use their definitions in equation (3.15),

$$Y(\rho) = \lambda \qquad \bar{Y}(\rho) = \bar{\lambda} \tag{4.5}$$

and obtain the integrability condition for these two equations by the invariant cross-differentiation by $\bar{Y}$ and Y using their commutation relation from equation (3.14):

$$F(Y(\bar{\lambda}) - \bar{Y}(\lambda)) = (u_t\rho + \tau)(\lambda - \bar{\lambda}). \tag{4.6}$$

This equation contains τ, so we use its definition (3.11)

$$\delta(\rho) = \tau. \tag{4.7}$$

Using the invariant cross-differentiation with Y or $\bar{Y}$ and δ, we obtain the compatibility conditions of equation (4.7) with each of equations (4.5):

$$\delta(\lambda) = Y(\tau) + 2u_t\lambda - \kappa\lambda^2 \tag{4.8}$$

and

$$\delta(\bar{\lambda}) = \bar{Y}(\tau) + 2u_t\bar{\lambda} - \kappa\bar{\lambda}^2. \tag{4.9}$$

These are complex conjugate to each other. There is one more differential consequence of the obtained resolving equations. This is the integrability condition of equation (4.6) solved with respect to $Y(\bar{\lambda})$ together with equation (4.9). It is obtained by the invariant cross-differentiation of these equations by δ and Y. Using the other resolving equations it can be brought to the form

$$F(Y(\bar{\lambda}) + \bar{Y}(\lambda)) = -(u_t\rho + \tau)(\lambda + \bar{\lambda}) + 2\kappa\left[\delta(\tau) + 2F + 4u_t\tau + \kappa\rho^2 + 2u_t^2\rho\right]. \tag{4.10}$$

The resolving equations (4.4), (4.6), (4.8), (4.9) and (4.10) form a closed resolving system if we assume that not only the second-order differential invariant $\eta = F$ but also

the third-order differential invariants $\lambda, \bar{\lambda}$ and τ are functions of t, u_t, ρ. They should be regarded as additional unknowns in these equations, so the resolving system consists of five partial differential equations with four unknowns $F, \lambda, \bar{\lambda}$ and τ. The operators of invariant differentiation are projected on the solution manifold of the heavenly equation and on the space of differential invariants treated as new independent variables. We keep the same notation for the projected operators of invariant differentiation and write them in the form

$$\delta = \partial_t + \left(\kappa\rho - u_t^2\right)\partial_{u_t} + \tau\partial_\rho \qquad Y = \partial_{u_t} + \lambda\partial_\rho \qquad \bar{Y} = \partial_{u_t} + \bar{\lambda}\partial_\rho. \qquad (4.11)$$

Here we have used the following properties of these operators

$$\delta(t) = 1 \qquad \delta(u_t) = \kappa\rho - u_t^2 \qquad \delta(\rho) = \tau$$

$$Y(t) = \bar{Y}(t) = 0 \qquad Y(u_t) = \bar{Y}(u_t) = 1 \qquad Y(\rho) = \lambda \qquad \bar{Y}(\rho) = \bar{\lambda} \qquad (4.12)$$

which follow from their definitions, equations (3.10), (3.11), (3.15) and the heavenly equation in the form (4.3). If we used formulae (4.11) for the operators of invariant differentiation $\delta, Y, \bar{Y}$ in the resolving equations (4.4), (4.6), (4.8), (4.9) and (4.10), then we would obtain the resolving system in an explicit form as a system of five first-order PDEs with four unknowns $F, \lambda, \bar{\lambda}, \tau$ and three independent variables t, u_t, ρ. This system is passive, i.e. it has no further algebraically independent first-order integrability conditions.

The commutator relations (3.14) were satisfied identically by the operators of invariant differentiation. In contrast, for the projected operators (4.11) these commutation relations and even the Jacobi identity

$$[\delta, [Y, \bar{Y}]] + [Y, [\bar{Y}, \delta]] + [\bar{Y}, [\delta, Y]] = 0 \qquad (4.13)$$

are not identically satisfied, but only on account of the resolving equations. It is easy to check that an even stronger statement is valid.

Theorem 1. *The commutator algebra (3.14) of the operators of invariant differentiation $\delta, Y, \bar{Y}$, together with the Jacobi identity (4.13), is equivalent to the resolving system for the heavenly equation and hence provides a commutator representation for this system.*

This theorem means that the complete set of resolving equations is encoded in the commutator algebra of the operators of invariant differentiation and provides the easiest way to derive the resolving system. In section 6 we shall show how the commutator representation of the resolving system can lead to a useful ansatz for solving this system.

5. Invariant and non-invariant solutions

Invariant solutions are defined as solutions that are invariant with respect to a symmetry subgroup of the equation. *Non-invariant solutions* are those solutions which are not invariant with respect to any one-parameter symmetry group of the equation. We present here a simple derivation of the infinitesimal criterion of invariance of solutions.

Consider a general form of the generator of a one-parameter symmetry of the heavenly equation as a linear combination of symmetry generators (2.2) with arbitrary real constant coefficients α and β,

$$X = \alpha\partial_t + \beta(t\partial_t + 2\partial_u) + a(z)\partial_z + \bar{a}(\bar{z})\partial_{\bar{z}} - (a'(z) + \bar{a}'(\bar{z}))\partial_u \qquad (5.1)$$

where $a(z)$ is an arbitrary holomorphic function. The infinitesimal criterion for the invariance of the solution $u = f(z, \bar{z}, t)$ with respect to the generator X has the general form (see para. 19.2.1 of [3])

$$X(f - u)|_{u=f} = 0 \qquad (5.2)$$

which for X defined by the formula (5.1) becomes

$$(\alpha + \beta t) f_t + a(z) f_z + \bar{a}(\bar{z}) f_{\bar{z}} = 2\beta - a'(z) - \bar{a}'(\bar{z}). \tag{5.3}$$

The invariance criterion can be summed up as follows.

Proposition 1. *If there exists a holomorphic function $a(z)$ and constants α and β, not all equal to zero, such that the equation (5.3) is satisfied, then the solution $u = f(z, \bar{z}, t)$ is invariant. Otherwise this solution is non-invariant.*

From this proposition one can derive some necessary conditions for the non-invariance of solutions. For example, we consider the case when $\alpha = 0$ and $\beta = 0$ so that equation (5.3) is a criterion of conformal invariance. The general solution of equation (5.3) in this case has the form

$$u = \ln f(\xi, t) - \ln a(z) - \ln \bar{a}(\bar{z}) \tag{5.4}$$

where

$$\xi = i \left(\int \frac{dz}{a(z)} - \int \frac{d\bar{z}}{\bar{a}(\bar{z})} \right). \tag{5.5}$$

The invariant ρ, defined by equation (2.7), becomes

$$\rho = \frac{f f_{\xi\xi} - f_\xi^2}{f^3} \tag{5.6}$$

and the invariants σ and $\bar{\sigma}$, defined by equations (3.11) as

$$\sigma = e^{-u} u_{\bar{z}t} D_z(\rho) \qquad \bar{\sigma} = e^{-u} u_{zt} D_{\bar{z}}(\rho)$$

are equal to each other

$$\bar{\sigma} = \sigma = \frac{(f f_{t\xi} - f_t f_\xi)}{f^3} \times \left(\frac{f f_{\xi\xi} - f_\xi^2}{f^3} \right)_\xi \tag{5.7}$$

where the subscripts denote partial differentiations. Hence the necessary condition for a solution to be conformally invariant is the equality

$$\bar{\sigma} = \sigma \quad (\Longleftrightarrow \bar{\lambda} = \lambda). \tag{5.8}$$

The converse statement gives the criterion for a solution to be conformally non-invariant.

Corollary 1. *The sufficient condition for a solution of the heavenly equation to be conformally non-invariant is that the following inequality should be satisfied:*

$$\bar{\sigma} \neq \sigma \tag{5.9}$$

(or equivalently $\bar{\lambda} \neq \lambda$).

Concerning the practical use of this statement we must remark that even if the inequality (5.9) is satisfied for a solution of the resolving system it could become the equality (5.8) on the corresponding solution of the heavenly equation. Nevertheless, the above criterion is useful, meaning that we should avoid solutions of the resolving equations satisfying equation (5.8) in order not to end up with conformally invariant solutions.

6. Particular solutions of the resolving system

Here we show that the commutator representation of the resolving system can prompt ansatzes, leading to particular solutions of the resolving equations. Attempts to solve the commutation relations by imposing relations between the operators of invariant differentiation lead to invariant solutions of the heavenly equation. This is the case with the ansatz $\bar{Y} = Y$. Then the expressions (4.11) for Y, $\bar{Y}$ imply $\bar{\lambda} = \lambda$, so the condition (5.9) of corollary 1 is not satisfied. Hence, we obtain a conformally invariant solution of the heavenly equation.

Another possible simplifying ansatz is that the operators Y and $\bar{Y}$ commute and we have

$$\tau = -u_t \rho \Rightarrow [Y, \bar{Y}] = 0 \tag{6.1}$$

but $\bar{Y} \neq Y$.

Before solving the resolving system with the ansatz (6.1) we keep in mind that $F \neq 0$. Indeed the case $F = 0$ is singular for the derivation of the resolving equations and should be treated separately. We shall consider first the case $F = 0$ and show that it leads to invariant solutions of the heavenly equation.

Putting $F = 0$ in equation (4.1), we obtain

$$u_{zt} = 0 \qquad u_{\bar{z}t} = 0 \tag{6.2}$$

and hence we have the separation

$$u = \alpha(t) + \beta(z, \bar{z}). \tag{6.3}$$

Substituting this expression for u into the heavenly equation (2.1), we obtain

$$e^{\alpha(t)}(\alpha''(t) + (\alpha'(t))^2) = \kappa e^{-\beta(z,\bar{z})}\beta_{z\bar{z}}(z, \bar{z}) = 2l \tag{6.4}$$

where $l = \bar{l}$ is a separation constant and the primes denote derivatives in t. Integrating the equation for α, we obtain

$$\alpha(t) = \ln\left(lt^2 + C_1 t + C_2\right) \tag{6.5}$$

where C_1, C_2 are arbitrary real constants. The equation for $\beta(z, \bar{z})$,

$$\beta_{z\bar{z}} = 2\kappa l e^{\beta} \tag{6.6}$$

is the Liouville equation if $\kappa = 1$ and the 'pseudo-Liouville' equation for $\kappa = -1$. Its general solution is

$$\beta(z, \bar{z}) = \ln a'(z) + \ln \bar{a}'(\bar{z}) - 2\ln(a(z) + \bar{a}(\bar{z})) - \ln l \tag{6.7}$$

if $\kappa = 1$ and

$$\beta(z, \bar{z}) = \ln a'(z) + \ln \bar{a}'(\bar{z}) - 2\ln(a(z)\bar{a}(\bar{z}) + 1) - \ln l \tag{6.8}$$

if $\kappa = -1$. Here $a(z)$ is an arbitrary holomorphic function and the primes denote derivatives. Thus, the corresponding solutions of the heavenly equation are given by equation (6.3) with $\alpha(t)$ determined by the formula (6.5) and $\beta(z, \bar{z})$ determined by formula (6.7) or (6.8).

To obtain the simplest representative of the orbit of solutions, we apply simplifying symmetry transformations: the conformal transformation

$$a(z) \mapsto z \qquad \bar{a}(\bar{z}) \mapsto \bar{z}$$

a suitable time translation and the dilation of time accompanied by a shift of u,

$$u \mapsto u + \ln l \qquad t \mapsto \frac{t}{\sqrt{l}}.$$

The resulting solutions become

$$u = \ln(t^2 + C) - 2\ln(z + \bar{z}) \qquad \text{if} \quad \kappa = 1 \tag{6.9}$$

$$u = \ln(t^2 + C) - 2\ln(z\bar{z} + 1) \qquad \text{if} \quad \kappa = -1 \tag{6.10}$$

where C is an arbitrary real constant.

To perform a check of invariance of the solutions (6.9) and (6.10), we substitute them into the criterion of invariance (5.3) and make a splitting in t. Then we obtain $\alpha = 0$ and if $C \neq 0$, then also $\beta = 0$. For $C = 0$ the constant β can be arbitrary. We also obtain a differential equation for $a(z)$ and $\bar{a}(\bar{z})$

$$a'(z) + \bar{a}'(\bar{z}) = 2\frac{a(z) + \bar{a}(\bar{z})}{z + \bar{z}} \qquad \text{for} \quad \kappa = 1 \tag{6.11}$$

and

$$a'(z) + \bar{a}'(\bar{z}) = 2\frac{\bar{z}a(z) + z\bar{a}(\bar{z})}{z\bar{z} + 1} \qquad \text{for} \quad \kappa = -1 \tag{6.12}$$

with the trivial solutions

$$a = \mathrm{i} \quad \bar{a} = -\mathrm{i} \quad \text{for } \kappa = 1 \qquad a = \mathrm{i}z \quad \bar{a} = -\mathrm{i}z \quad \text{for } \kappa = -1. \tag{6.13}$$

Thus, we have proved that there exist α, β and $a(z), \bar{a}(\bar{z})$ such that the criterion (5.3) of invariance of solutions is satisfied for our solutions (6.9) and (6.10). Hence the case $F = 0$ corresponds to invariant solutions.

In the following we assume that $F \neq 0$ and consider the resolving equations with the ansatz (6.1). Equations (4.6) and (4.10) become, respectively,

$$Y(\bar{\lambda}) = \bar{Y}(\lambda) \qquad \text{and} \qquad Y(\bar{\lambda}) + \bar{Y}(\lambda) = 4\kappa$$

and hence

$$\bar{Y}(\lambda) = 2\kappa \qquad Y(\bar{\lambda}) = 2\kappa. \tag{6.14}$$

Next we consider the compatibility condition of the system of equations (4.8) and (4.9) and the first equation in (6.14). Because of the formula (6.1) the first equation becomes

$$\delta(\lambda) = u_t\lambda - \kappa\lambda^2 - \rho. \tag{6.15}$$

Then, using cross-differentiation of the invariant operators δ and $\bar{Y}$, their commutator (3.14) and equation (6.14), we obtain a very simple result

$$\lambda + \bar{\lambda} = 2\kappa u_t. \tag{6.16}$$

Solving this equation with respect to $\bar{\lambda}$, substituting in equation (4.9) and using equation (6.15) to express $\delta(\lambda)$, we obtain a quadratic equation for λ,

$$\lambda^2 - 2\kappa u_t\lambda + 2\kappa\rho = 0$$

with the solution

$$\lambda = \kappa u_t + \mathrm{i}\sqrt{2\kappa\rho - u_t^2} \tag{6.17}$$

where we have chosen the + sign before the square root. Equation (6.16) gives the result

$$\bar{\lambda} = \kappa u_t - \mathrm{i}\sqrt{2\kappa\rho - u_t^2} \tag{6.18}$$

which is complex conjugate to (6.17) provided the condition

$$2\kappa\rho - u_t^2 \geqslant 0 \tag{6.19}$$

is satisfied. The obtained expressions for λ and $\bar{\lambda}$ satisfy equations (6.14), (6.15) and its complex conjugate, and hence all the resolving equations apart from equation (4.4). We rewrite this last equation for the new unknown $f = \ln F$ as

$$f_t + \left(\kappa\rho - u_t^2\right) f_{u_t} - u_t\rho f_\rho = -3u_t \tag{6.20}$$

and solve it by the method of characteristics obtaining the general solution of equation (4.4)

$$F = \rho^3\varphi(\xi, \theta) \qquad \text{where} \quad \xi = \frac{2\kappa\rho - u_t^2}{\rho^2} \quad \theta = t - \frac{\kappa}{\rho}\left(u_t + \sqrt{2\kappa\rho - u_t^2}\right) \tag{6.21}$$

where φ is an arbitrary real differentiable function.

Finally, we sum up our results for the particular solution of the resolving system which follows from our ansatz (6.1):

$$
\begin{aligned}
F &= \rho^3\varphi(\xi, \theta) & \tau &= -u_t\rho \\
\lambda &= \kappa u_t + i\sqrt{2\kappa\rho - u_t^2} & \bar{\lambda} &= \kappa u_t - i\sqrt{2\kappa\rho - u_t^2}.
\end{aligned}
\tag{6.22}
$$

This will be used in the next section for obtaining the corresponding solution of the heavenly equation. Since $\bar{\lambda} \neq \lambda$ the condition (5.9) of corollary 1 for non-invariance of this solution is satisfied.

7. Invariant integration and non-invariant solution of the heavenly equation

In this section we reconstruct the solution of the heavenly equation starting from the particular solution (6.22) of the resolving system. We demonstrate here the procedure of *invariant integration* which amounts to the transformation of equations to the form of the exact invariant derivative. Then we drop the operator of invariant differentiation adding the term playing the role of the integration constant which is an arbitrary element of the kernel of this operator.

We start from our ansatz (6.1) using the definitions $\tau = \delta(\rho)$ and $\delta = D_t$,

$$D_t(\ln\rho) = D_t(-u). \tag{7.1}$$

We integrate this equation in the form

$$\ln\rho = -u + \ln\gamma_{z\bar{z}}(z, \bar{z})$$

where the last term is a function to be determined. Solving this equation with respect to ρ and using the definition of ρ, we obtain

$$\rho = e^{-u}u_{z\bar{z}} = e^{-u}\gamma_{z\bar{z}}(z, \bar{z})$$

and hence $u_{z\bar{z}} = \gamma_{z\bar{z}}(z, \bar{z})$. This implies the following form of the solution:

$$u(z, \bar{z}, t) = \gamma(z, \bar{z}) + \alpha(z, t) + \bar{\alpha}(\bar{z}, t) \tag{7.2}$$

where γ, α and $\bar{\alpha}$ are arbitrary smooth functions of two variables. After the substitution of this expression into the heavenly equation (2.1), it becomes

$$e^{\alpha(z,t)+\bar{\alpha}(\bar{z},t)}\left[\alpha_{tt}(z, t) + \bar{\alpha}_{tt}(\bar{z}, t) + (\alpha_t(z, t) + \bar{\alpha}_t(\bar{z}, t))^2\right] = \kappa e^{-\gamma(z,\bar{z})}\gamma_{z\bar{z}}(z, \bar{z}). \tag{7.3}$$

Next we rewrite the formulae (6.22) for λ and $\bar{\lambda}$ in the form of exact invariant derivatives:

$$Y\left(\sqrt{2\kappa\rho - u_t^2} - i\kappa u_t\right) = 0 \qquad \bar{Y}\left(\sqrt{2\kappa\rho - u_t^2} + i\kappa u_t\right) = 0. \tag{7.4}$$

On account of the definitions (3.13), the operators Y and $\bar{Y}$ can be written as

$$Y = \frac{1}{F}\Delta = \frac{e^{-u}u_{\bar{z}t}}{F}D_z \qquad \bar{Y} = \frac{1}{F}\bar{\Delta} = \frac{e^{-u}u_{zt}}{F}D_{\bar{z}}$$

and equations (7.4) become

$$\left(\sqrt{2\kappa\rho - u_t^2} + i\kappa u_t\right)_{\bar{z}} = 0 \qquad \left(\sqrt{2\kappa\rho - u_t^2} - i\kappa u_t\right)_z = 0.$$

They are integrated in the form

$$\sqrt{2\kappa\rho - u_t^2} + i\kappa u_t = \psi(z, t) \qquad \sqrt{2\kappa\rho - u_t^2} - i\kappa u_t = \bar{\psi}(\bar{z}, t) \qquad (7.5)$$

where $\psi, \bar{\psi}$ are arbitrary smooth functions. Taking the difference of two equations (7.5), we obtain

$$u_t = -\frac{i\kappa}{2}[\psi(z, t) - \bar{\psi}(\bar{z}, t)] = \alpha_t(z, t) + \bar{\alpha}_t(\bar{z}, t)$$

where the last equality follows from the expression (7.2) for u. Separation of $z, \bar{z}$ in the last equality leads to

$$\alpha_t(z, t) + \frac{i\kappa}{2}\psi(z, t) = -\left[\bar{\alpha}_t(\bar{z}, t) - \frac{i\kappa}{2}\bar{\psi}(\bar{z}, t)\right] = \chi'(t) = -\bar{\chi}'(t)$$

where $\chi'(t)$ is the separation 'constant' and the prime denotes the derivative. Solving these equations with respect to $\psi, \bar{\psi}$ and substituting the results into equations (7.5), we solve them with respect to the square root with the result

$$\sqrt{2\kappa\rho - u_t^2} = i\kappa[\alpha_t(z, t) - \bar{\alpha}_t(\bar{z}, t) - 2\chi'(t)].$$

Solving this equation with respect to $\kappa\rho$ and multiplying the result by $e^{\alpha + \bar{\alpha}}$, we obtain

$$\kappa e^{-\gamma(z,\bar{z})}\gamma_{z\bar{z}}(z, \bar{z}) = 2e^{\alpha(z,t)+\bar{\alpha}(\bar{z},t)}\left[\alpha_t(z, t)\bar{\alpha}_t(\bar{z}, t) + \chi'(t)(\alpha_t(z, t) - \bar{\alpha}_t(\bar{z}, t)) - \chi'^2(t)\right].$$
$$(7.6)$$

Using this equation on the right-hand side of the heavenly equation in the form (7.3) and separating $z, \bar{z}$, we obtain two complex conjugate equations

$$\alpha_{tt}(z, t) = -\alpha_t^2(z, t) + 2\chi'(t)\alpha_t(z, t) - \chi'^2(t) + \mu(t) \qquad (7.7)$$

$$\bar{\alpha}_{tt}(\bar{z}, t) = -\bar{\alpha}_t^2(\bar{z}, t) - 2\chi'(t)\bar{\alpha}_t(\bar{z}, t) - \chi'^2(t) - \mu(t) \qquad (7.8)$$

where $\mu(t) = -\bar{\mu}(t)$ is the separation 'constant'. We substitute these expressions for α_{tt} and $\bar{\alpha}_{tt}$ into the transformed heavenly equation (7.3) to obtain

$$e^{\alpha(z,t)+\bar{\alpha}(\bar{z},t)}\left[\alpha_t(z, t)\bar{\alpha}_t(\bar{z}, t) + \chi'(t)(\alpha_t(z, t) - \bar{\alpha}_t(\bar{z}, t) - \chi'^2(t))\right] = \frac{\kappa}{2}e^{-\gamma(z,\bar{z})}\gamma_{z\bar{z}}(z, \bar{z}).$$
$$(7.9)$$

Next we take the total derivative D_t of this equation and substitute again the second derivatives α_{tt} and $\bar{\alpha}_{tt}$ from equations (7.7) and (7.8). The result is unexpectedly simple:

$$(\chi'' - \mu)(\alpha_t - \bar{\alpha}_t - 2\chi') = 0. \qquad (7.10)$$

This equation implies that

$$\mu(t) = \chi''(t) \qquad (7.11)$$

since the complementary assumption

$$\alpha_t - \bar{\alpha}_t - 2\chi' = 0$$

leads again to equation (7.11). Indeed, the last equation allows a separation of $z, \bar{z}$ and, being integrated, gives $\alpha, \bar{\alpha}$:

$$\alpha(z, t) = \chi(t) + \nu(t) + \omega(z) \qquad \bar{\alpha}(\bar{z}, t) = -\chi(t) + \nu(t) + \bar{\omega}(\bar{z}).$$

Substituting these expressions into equations (7.7) and (7.8) and comparing the results, we discover again equation (7.11).

With this restriction equations (7.7) and (7.8) are simplified and integrated to give

$$\alpha(z, t) = \ln(t + b(z)) + \chi(t) + \omega(z) \qquad \bar{\alpha}(\bar{z}, t) = \ln(t + \bar{a}(\bar{z})) - \chi(t) + \bar{\omega}(\bar{z}) \qquad (7.12)$$

where $b(z)$ and $\omega(z)$ are arbitrary holomorphic functions and we have reserved the notation $a(z)$ only for the generators of the conformal vector field X_a in equation (2.2).

Now define a new function of $z, \bar{z}$,

$$\Gamma(z, \bar{z}) = \gamma(z, \bar{z}) + \omega(z) + \bar{\omega}(\bar{z}) \qquad (7.13)$$

so that the form (7.2) of the solution becomes

$$u(z, \bar{z}, t) = \ln(t + b(z)) + \ln(t + \bar{b}(\bar{z})) + \Gamma(z, \bar{z}). \qquad (7.14)$$

Substituting the expressions (7.12) for $\alpha, \bar{\alpha}$ into the transformed heavenly equation (7.3), we obtain the equation for the only unknown function $\Gamma(z, \bar{z})$ in the solution (7.14):

$$\Gamma_{z\bar{z}} = 2\kappa e^{\Gamma}. \qquad (7.15)$$

If $\kappa = 1$, this is the Liouville equation with the general solution

$$\Gamma(z, \bar{z}) = \ln c'(z) + \ln \bar{c}'(\bar{z}) - 2\ln(c(z) + \bar{c}(\bar{z})) \qquad (7.16)$$

where $c(z)$ is an arbitrary holomorphic function. If $\kappa = -1$, we call equation (7.15) the 'pseudo-Liouville' equation and its general solution is

$$\Gamma(z, \bar{z}) = \ln c'(z) + \ln \bar{c}'(\bar{z}) - 2\ln(c(z)\bar{c}(\bar{z}) + 1). \qquad (7.17)$$

Finally, substituting these expressions for $\Gamma(z, \bar{z})$ into equation (7.14) we obtain the solutions of the heavenly equation (2.1) for the two choices of the sign $\kappa = +1$ and $\kappa = -1$.

1. The solution for $\kappa = 1$:

$$u(z, \bar{z}, t) = \ln(t + b(z)) + \ln(t + \bar{b}(\bar{z})) + \ln c'(z) + \ln \bar{c}'(\bar{z}) - 2\ln(c(z) + \bar{c}(\bar{z})). \qquad (7.18)$$

2. The solution for $\kappa = -1$ (see also [16]):

$$u(z, \bar{z}, t) = \ln(t + b(z)) + \ln(t + \bar{b}(\bar{z})) + \ln c'(z) + \ln \bar{c}'(\bar{z}) - 2\ln(c(z)\bar{c}(\bar{z}) + 1). \qquad (7.19)$$

Here $b(z)$ and $c(z)$ are arbitrary holomorphic functions.

To avoid 'false generality' it is sufficient to choose the simplest representative of the obtained orbits of solutions applying the conformal symmetry transformation $c(z) = z$, $\bar{c}(\bar{z}) = \bar{z}$ with the following results.

1. The solution for $\kappa = 1$:

$$u(z, \bar{z}, t) = \ln(t + b(z)) + \ln(t + \bar{b}(\bar{z})) - 2\ln(z + \bar{z}). \qquad (7.20)$$

2. The solution for $\kappa = -1$:

$$u(z, \bar{z}, t) = \ln(t + b(z)) + \ln(t + \bar{b}(\bar{z})) - 2\ln(z\bar{z} + 1). \qquad (7.21)$$

Here $b(z)$ is still an arbitrary holomorphic function.

Up to now we solved completely only the ansatz (6.1) defining τ, but we did not check the automorphic equation (4.1) and the auxiliary equations (4.5), by using the particular solution (6.22) of the resolving system. Hence, though we obtained the correct solutions (7.20) and (7.21) of the heavenly equation (2.1), we have not made a complete foliation of these solutions into separate orbits.

To do this, first we remark that due to the discrete symmetry of the heavenly equation (2.1) and of our solutions with respect to the permutation $z \leftrightarrow \bar{z}$, we can define the holomorphic function $b(z)$ as satisfying the condition

$$\text{Im}\, b(z) \geqslant 0 \quad \text{for}\ \kappa = 1 \qquad \text{and} \qquad \text{Im}\, b(z) \leqslant 0 \quad \text{for}\ \kappa = -1. \tag{7.22}$$

Then we check that the automorphic equation (4.1) coincides with the auxiliary equations (4.5) and becomes

$$(z + \bar{z})^2 b'(z) \bar{b}'(\bar{z}) = 8\varphi(\xi, \theta) \qquad \text{for}\ \ \kappa = 1 \tag{7.23}$$

and

$$(z\bar{z} + 1)^2 b'(z) \bar{b}'(\bar{z}) = -8\varphi(\xi, \theta) \qquad \text{for}\ \ \kappa = -1. \tag{7.24}$$

Using the solutions (7.20) and (7.21) in the definitions (6.21) of the characteristic variables ξ and θ, we discover that they depend only on b and $\bar{b}$, i.e.

$$\xi = -\frac{(b - \bar{b})^2}{4} \qquad \theta = -\frac{(b + \bar{b})}{2} - \sqrt{\xi}. \tag{7.25}$$

Hence, defining the new arbitrary function $\Phi(b, \bar{b}) = \varphi(\xi, \theta)$, the automorphic equations (7.23) and (7.24) become

$$(z + \bar{z})^2 b'(z) \bar{b}'(\bar{z}) = 8\Phi(b, \bar{b}) \qquad \text{for}\ \ \kappa = 1 \tag{7.26}$$

and

$$(z\bar{z} + 1)^2 b'(z) \bar{b}'(\bar{z}) = -8\Phi(b, \bar{b}) \qquad \text{for}\ \ \kappa = -1. \tag{7.27}$$

Sufficient conditions for solving these functional-differential equations are given by the following choices of $\Phi(b, \bar{b})$:

$$\Phi(b, \bar{b}) = \frac{[f(b) + \bar{f}(\bar{b})]^2}{8 f'(b) \bar{f}'(\bar{b})} \qquad \text{for}\ \ \kappa = 1 \tag{7.28}$$

and

$$\Phi(b, \bar{b}) = -\frac{[f(b) \bar{f}(\bar{b}) + 1]^2}{8 f'(b) \bar{f}'(\bar{b})} \qquad \text{for}\ \ \kappa = -1 \tag{7.29}$$

where $f(b)$ is an arbitrary holomorphic function. Then the automorphic equations become

$$\{\ln[f(b) + \bar{f}(\bar{b})]\}_{z\bar{z}} = [\ln(z + \bar{z})]_{z\bar{z}} \qquad \text{for}\ \ \kappa = 1 \tag{7.30}$$

and

$$\{\ln[f(b) \bar{f}(\bar{b}) + 1]\}_{z\bar{z}} = [\ln(z\bar{z} + 1)]_{z\bar{z}} \qquad \text{for}\ \ \kappa = -1. \tag{7.31}$$

Their general solutions are

$$f(b) + \bar{f}(\bar{b}) = w(z)\bar{w}(\bar{z})(z + \bar{z}) \qquad \text{for}\ \ \kappa = 1 \tag{7.32}$$

and

$$f(b) \bar{f}(\bar{b}) + 1 = w(z)\bar{w}(\bar{z})(z\bar{z} + 1) \qquad \text{for}\ \ \kappa = -1 \tag{7.33}$$

where $w(z)$ is an arbitrary holomorphic function. The formulae (7.28) and (7.29) for $\Phi(b, \bar{b})$ become

$$\Phi(b, \bar{b}) = \frac{w^2(z) \bar{w}^2(\bar{z})(z + \bar{z})^2}{8 f'(b) \bar{f}'(\bar{b})} \qquad \text{for}\ \ \kappa = 1$$

and

$$\Phi(b, \bar{b}) = -\frac{w^2(z) \bar{w}^2(\bar{z})(z\bar{z} + 1)^2}{8 f'(b) \bar{f}'(\bar{b})} \qquad \text{for}\ \ \kappa = -1.$$

If we plug these formulae into the automorphic equations (7.26) and (7.27), both automorphic equations coincide and become

$$b'(z)\bar{b}'(\bar{z}) = \frac{w^2(z)\bar{w}^2(\bar{z})}{f'(b)\bar{f}'(\bar{b})}. \tag{7.34}$$

This equation admits a separation of variables, leading to the ODEs

$$b'(z) = \frac{w^2(z)}{f'(b)} \qquad \bar{b}'(z) = \frac{\bar{w}^2(\bar{z})}{\bar{f}'(\bar{b})}. \tag{7.35}$$

The obvious choice of the functions $w(z)$ and $\bar{w}(\bar{z})$,

$$w(z) = 1 \Longleftrightarrow \bar{w}(\bar{z}) = 1 \tag{7.36}$$

simplifies the ODEs (7.35) to

$$[f(b)]_z = 1 \qquad [\bar{f}(\bar{b})]_{\bar{z}} = 1 \tag{7.37}$$

with the solution

$$f[b(z)] = z \qquad \bar{f}[\bar{b}(\bar{z})] = \bar{z} \tag{7.38}$$

meaning that $b(z)$ is the inverse function for $f(b)$: $b = f^{-1}$. The equations (7.32) and (7.33) are obviously satisfied by the solution (7.38) with our choice (7.36) of $w(z)$, $\bar{w}(\bar{z})$.

Thus, any particular function $b(z)$ can be obtained for an appropriate choice of $f(b)$ as its inverse function. This fixes the function $\Phi(b, \bar{b})$ according to the formula (7.28) or (7.29), the function $\varphi(\xi, \theta) = \Phi(b, \bar{b})$ and the right-hand side F of the automorphic equation (4.1) determined by the formulae (6.22). Hence, any particular choice of the function $b(z)$ in our solutions (7.20) and (7.21) means a corresponding choice of the particular orbit in the solution space of the heavenly equation.

8. Check of non-invariance of the solutions

In this section we prove that our solutions (7.20) and (7.21) of the heavenly equation (2.1) are non-invariant, with respect to its symmetry group generated by the vector fields in (2.2), for generic functions $a(z)$, except for some particular classes listed below in the theorems summarizing the results.

For the check of non-invariance we substitute our solutions (7.20) and (7.21) into the invariance criterion (5.3). The resulting equation is quadratic in t and it implies the vanishing of the coefficients of t^2, t and t^0.

We consider first the case $\kappa = 1$. The term with t^2 gives again equation (6.11) of section 6. However, now we need the general solution of this equation.

We assume in the generic case that $a'(z) + \bar{a}'(\bar{z}) \neq 0$, otherwise equation (6.11) implies $a = -\bar{a} =$ constant and this case should be treated separately. Then we rewrite equation (6.11) in the form

$$\frac{a(z) + \bar{a}(\bar{z})}{a'(z) + \bar{a}'(\bar{z})} = \frac{z + \bar{z}}{2} \Longrightarrow \left(\frac{a + \bar{a}}{a' + \bar{a}'}\right)_{z\bar{z}} = 0. \tag{8.1}$$

In order to consider the generic case we postulate $a''\bar{a}'' \neq 0$, then the last equation can be easily manipulated, obtaining the solution

$$a(z) = C_1(z + \lambda)^2 + C_2 \qquad \bar{a}(\bar{z}) = -\left[C_1(\bar{z} - \lambda)^2 + C_2\right] \tag{8.2}$$

where $C_1 \neq 0$, C_2 and λ are arbitrary purely imaginary constants.

Now we consider the term without t in the criterion of invariance using our result (8.2), which gives the equation with the separated variables $z, \bar{z}$,

$$\frac{\alpha}{b(z)} + \left[C_1(z+\lambda)^2 + C_2\right]\frac{b'(z)}{b(z)} - \beta$$

$$= -\left\{\frac{\alpha}{\bar{b}(\bar{z})} - \left[C_1(\bar{z}-\lambda)^2 + C_2\right]\frac{\bar{b}'(\bar{z})}{\bar{b}(\bar{z})} - \beta\right\} = \mu = -\bar{\mu} \qquad (8.3)$$

where μ is a separation constant. Comparing these equations with the equation obtained from the term with t in the criterion of invariance, we conclude that they coincide if and only if the condition

$$\mu(b(z) - \bar{b}(\bar{z})) = 0$$

is satisfied. This implies $\mu = 0$, since otherwise we have $b = \bar{b} = $ constant and our solution is obviously invariant depending on only two variables t and $z + \bar{z}$. Hence equations (8.3) become

$$\left[C_1(z+\lambda)^2 + C_2\right]b'(z) - \beta b(z) = -\alpha \qquad (8.4)$$

$$\left[C_1(\bar{z}-\lambda)^2 + C_2\right]\bar{b}'(\bar{z}) + \beta\bar{b}(\bar{z}) = \alpha. \qquad (8.5)$$

Consider now the case $C_2 \neq 0$, $\beta \neq 0$ and introduce the notation

$$\nu = \sqrt{-\frac{C_2}{C_1}} \qquad \gamma = \frac{\beta}{2\sqrt{-C_1C_2}}. \qquad (8.6)$$

Integrating the ODEs (8.4) and (8.5), we fix the function $b(z)$ in our solution of the heavenly equation which corresponds to the invariant solution in the considered case,

$$b(z) = C\left(\frac{z+\lambda-\nu}{z+\lambda+\nu}\right)^\gamma + \frac{\alpha}{\beta} \qquad \bar{b}(\bar{z}) = \bar{C}\left(\frac{\bar{z}-\lambda-\nu}{\bar{z}-\lambda+\nu}\right)^\gamma + \frac{\alpha}{\beta} \qquad (8.7)$$

where $C, \bar{C}$ are integration constants.

In a similar way we treat other possible cases. We sum up the results for the case of $\kappa = 1$ in the following statement.

Theorem 2. *The function*

$$u = \ln(t + b(z)) + \ln(t + \bar{b}(\bar{z})) - 2\ln(z + \bar{z}) \qquad (8.8)$$

is a solution of the heavenly equation (2.1) for $\kappa = +1$ for an arbitrary holomorphic function $b(z)$. This solution is a non-invariant solution of this equation iff the function $b(z)$ does not coincide with any of the following choices:

1.

$$b(z) = C\left(\frac{z+\lambda-\nu}{z+\lambda+\nu}\right)^\gamma + \frac{\alpha}{\beta}$$

where α and β are arbitrary real constants, $\beta \neq 0$, ν and γ are defined by the formulae (8.6) and λ, C_1, C_2 are complex constants which satisfy the conditions

$$\bar{\lambda} = -\lambda \qquad \bar{C}_1 = -C_1 \qquad \bar{C}_2 = -C_2 \qquad C_1 \neq 0 \qquad C_2 \neq 0.$$

In this case the solution is invariant with respect to the symmetry generator

$$X = \alpha\partial_t + \beta(t\partial_t + 2\partial_u) + C_1\left[(z+\lambda)^2\partial_z - (\bar{z}-\lambda)^2\partial_{\bar{z}} - 2(z-\bar{z})\partial_u\right] + C_2\left(\partial_z - \partial_{\bar{z}}\right).$$

2.

$$b(z) = \frac{\alpha}{2\sqrt{-C_1 C_2}} \left(\frac{z + \lambda + \nu}{z + \lambda - \nu} \right) + C \qquad \text{if} \quad \beta = 0 \quad C_2 \neq 0.$$

The solution is invariant with respect to the previous symmetry generator X with $\beta = 0$.

3.

$$b(z) = C \exp \left[-\frac{\beta}{C_1(z + \lambda)} \right] + \frac{\alpha}{\beta} \qquad \text{if} \quad C_2 = 0 \quad \beta \neq 0.$$

The solution is invariant with respect to the symmetry generator X from case 1 with $C_2 = 0$.

4.

$$b(z) = \frac{\alpha}{C_1(z + \lambda)} + C \qquad \text{if} \quad \beta = 0 \text{ and } C_2 = 0.$$

The solution is invariant with respect to the symmetry generator X from case 1 with $\beta = 0$ and $C_2 = 0$.

5.

$$b(z) = C(C_1 z + C_2)^{\beta/C_1} + \frac{\alpha}{\beta} \qquad \text{if} \quad C_1 \neq 0 \quad \beta \neq 0.$$

The solution is invariant with respect to the symmetry generator

$$X = \alpha \partial_t + \beta \left(t \partial_t + 2 \partial_u \right) + C_1 \left(z \partial_z + \bar{z} \partial_{\bar{z}} - 2 \partial_u \right) + C_2 \left(\partial_z - \partial_{\bar{z}} \right).$$

6.

$$b(z) = C \mathrm{e}^{\frac{\beta}{C_2} z} + \frac{\alpha}{\beta} \qquad \text{if} \quad C_1 = 0 \quad \beta \neq 0.$$

The solution is invariant with respect to the symmetry generator X from case 5 with $C_1 = 0$.

7.

$$b(z) = -\frac{\alpha}{C_2} z + C \qquad \text{if} \quad C_1 = 0 \quad \beta = 0.$$

The solution is invariant with respect to the symmetry generator X from case 5 with $C_1 = 0$ and $\beta = 0$.

8.

$$b(z) = b = constant \qquad \text{if} \quad C_1 = \alpha = \beta = 0 \quad C_2 \neq 0.$$

The solution is invariant with respect to the symmetry generator

$$X = \partial_z - \partial_{\bar{z}}.$$

If $b = \alpha/\beta$, then this solution is also invariant with respect to the generator

$$X = \alpha \partial_t + \beta \left(t \partial_t + 2 \partial_u \right).$$

Now we consider the case $\kappa = -1$ and substitute the solution (7.21) of the heavenly equation (2.1) into the criterion of invariance (5.3). Then the resulting equation is again quadratic in t and the term with t^2 gives us again equation (6.12) for which we need now the general solution. First, we rewrite it in the form

$$a' + \bar{a}' + \frac{a' + \bar{a}'}{z\bar{z}} = 2 \left(\frac{a}{z} + \frac{\bar{a}}{\bar{z}} \right).$$

Differentiating this equation with respect to z and $\bar{z}$, we obtain an equation which admits separation of $z, \bar{z}$ in the form

$$za''(z) - a'(z) = -[\bar{z}\bar{a}''(\bar{z}) - \bar{a}'(\bar{z})] = \lambda = -\bar{\lambda} \tag{8.9}$$

where λ is a separation constant. Integrating these ODEs, we obtain

$$a(z) = C_1 z^2 - \lambda z + C_2 \qquad \bar{a}(\bar{z}) = \bar{C}_1 \bar{z}^2 + \lambda \bar{z} + \bar{C}_2$$

where C_1, C_2 are integration constants. Substituting these solutions into equation (6.12), we see that it is identically satisfied if and only if $\bar{C}_1 = C_2 \iff \bar{C}_2 = C_1$, so that finally we have the solution of the equation following from the term with t^2:

$$a(z) = C_1 z^2 - \lambda z + C_2 \qquad \bar{a}(\bar{z}) = C_2 \bar{z}^2 + \lambda \bar{z} + C_1. \qquad (8.10)$$

Next we consider the term without t in the criterion of invariance using our result (8.10), which gives the equation with the separated variables $z, \bar{z}$:

$$\frac{\alpha}{b(z)} + \left(C_1 z^2 - \lambda z + C_2\right) \frac{b'(z)}{b(z)} - \beta$$

$$= -\left[\frac{\alpha}{\bar{b}(\bar{z})} + \left(C_2 \bar{z}^2 + \lambda \bar{z} + C_1\right) \frac{\bar{b}'(\bar{z})}{\bar{b}(\bar{z})} - \beta\right] = \mu = -\bar{\mu} \qquad (8.11)$$

where μ is a separation constant. Comparing these equations with the equation obtained from the term with t in the criterion of invariance, we conclude that they coincide if and only if the condition

$$\mu(b(z) - \bar{b}(\bar{z})) = 0$$

is satisfied. This implies $\mu = 0$ for the same reason as in the case $\kappa = 1$. Hence, equations (8.11) become

$$\left(C_1 z^2 - \lambda z + C_2\right) b'(z) - \beta b(z) = -\alpha \qquad (8.12)$$

$$\left(C_2 \bar{z}^2 + \lambda \bar{z} + C_1\right) \bar{b}'(\bar{z}) - \beta \bar{b}(\bar{z}) = -\alpha. \qquad (8.13)$$

Consider now the case $C_1 \neq 0$. Introduce the new constants $\tilde{\lambda} = -\lambda/(2C_1)$ and $\tilde{C}_2 = C_2 - \lambda^2/(4C_1)$. Then the first equation takes the form

$$\left[C_1(z + \tilde{\lambda})^2 + \tilde{C}_2\right] b'(z) - \beta b(z) = -\alpha \qquad (8.14)$$

coinciding with the ODE (8.4) in the case $\kappa = 1$. Therefore we can use its solutions with an appropriate change of notation. Other possible cases are treated in a similar way. Therefore, we can transfer the results of theorem 2 to the case $\kappa = -1$ with an appropriate change of notation and sum them up in the following statement.

Theorem 3. *The function*

$$u = \ln(t + b(z)) + \ln(t + \bar{b}(\bar{z})) - 2\ln(z\bar{z} + 1) \qquad (8.15)$$

is a solution of the heavenly equation (2.1) for $\kappa = -1$ for an arbitrary holomorphic function $b(z)$. This solution is a non-invariant solution of this equation iff the function $b(z)$ does not coincide with any of the eight forms given in theorem 2 with the change of notation

$$\lambda \mapsto \tilde{\lambda} = -\frac{\lambda}{2C_1} \qquad C_2 \mapsto \tilde{C}_2 = C_2 - \frac{\lambda^2}{4C_1}$$

$$\nu \mapsto \tilde{\nu} = \sqrt{-\frac{\tilde{C}_2}{C_1}} \qquad \gamma \mapsto \tilde{\gamma} = \frac{\beta}{2\sqrt{-C_1 \tilde{C}_2}}$$

in cases $1, 2, 3, 4$ and $C_1 \mapsto -\lambda$ in case 5. Those eight choices of $b(z)$ give invariant solutions with respect to the corresponding symmetry generators of theorem 2 with the same change of notation.

9. Conclusions and outlook

The title of this paper, or rather of the research direction that it represents, could have been 'Invariant methods for obtaining non-invariant solutions of partial differential equations'. The main result is that we are proposing an alternative tool for obtaining particular solutions of nonlinear partial differential equations with infinite-dimensional symmetry algebras. As stated in the introduction, the idea of the method is more than a hundred years old [1, 2]. We have turned it into a usable tool by adding new elements. These are:

1. The systematic use of invariant cross-differentiation involving the operators of invariant differentiation and their commutator algebra for the derivation of the resolving equations and for obtaining their particular solutions.
2. The presentation of the resolving system as a Lie algebra of the operators of invariant differentiation (over the field of differential invariants of the symmetry group) [11].
3. The concept of invariant integration applied to the automorphic system.

Let us use the heavenly equation (2.1) to compare different methods of obtaining exact analytical solutions of a partial differential equation, provided or at least suggested by symmetry analysis. In all of them the studied equation is embedded into a larger system of equations, to be solved simultaneously.

The most standard method is that of invariant solutions [3, 14, 17]. One first finds the symmetry algebra realized by vector fields of the form

$$X = \tau \partial_t + \xi \partial_z + \bar{\xi} \partial_{\bar{z}} + \phi \partial_u \tag{9.1}$$

where $\tau, \xi, \bar{\xi}$ and ϕ are functions of $t, z, \bar{z}$ and u. Once this algebra is found (i.e. the algebra (2.2) for the heavenly equation), one classifies its subalgebras into conjugacy classes and then adds one, or more, first-order linear equations of the type

$$\tau u_t + \xi u_z + \bar{\xi} u_{\bar{z}} - \phi = 0 \tag{9.2}$$

to the studied equation. These equations are solved, and their solution is substituted into the original equation. This again is solved, and we obtain solutions invariant under the chosen subgroup.

Further methods are the Bluman and Cole 'non-classical method' [18], the Clarkson–Kruskal [19] 'direct method' and that of 'conditional symmetries' [20] (see [21] for a review). These methods, basically all equivalent, amount to the fact that a first-order equation of type (9.2) is added to the studied equation, without the requirement that $\tau, \xi, \bar{\xi}$ and ϕ define an element of the symmetry algebra.

Finally, we have the group foliation method [11] used and further developed in this paper. Let us review the essential steps, performed above.

1. Find the symmetry algebra (2.2).
2. Find all differential invariants of order up to N of the symmetry group, or one of its subgroups. In the present case this is the conformal group generated by X_a of equation (2.2). The number N must be larger or equal to the order of the equation and must satisfy the requirement that there should be $\#N$ functionally independent invariants with

$$\#N > p + q \tag{9.3}$$

where p and q are the numbers of independent and dependent variables, respectively. In our case we have $p = 3, q = 1, N = 2, \#N = 5$. The actual invariants are given in equation (2.7).

3. Choose p invariants as new independent variables and require that the remaining invariants be functions of the chosen ones. This provides us with the automorphic system that also contains the considered equation expressed in terms of the invariants. In our case the automorphic system consists of equation (2.8) (the heavenly equation) and equation (4.1) (or equivalently (4.2)).

4. Find the 'resolving equations'. This is a set of compatibility conditions between the studied equation and those that we have added to obtain the automorphic system. In our case we require compatibility between equations (2.8) and (4.1), i.e. determine the restrictions on the function $F(t, u_t, \rho)$. We have shown that this can be done in an explicitly invariant manner by using the operators of invariant differentiation, in our case δ, Y and $\bar{Y}$ of equations (3.9) and (3.13). The resolving system in our case consists of equations (4.4), (4.6), (4.8), (4.9) and (4.10). As stated by the fundamental theorem 1, this resolving system is best written as a system of commutator relations for the operators of invariant differentiation projected on the solution manifold of the heavenly equation in the space of differential invariants, together with the Jacobi relations for these operators.

5. Solve the resolving system and the automorphic one. This provides solutions of the original equation.

The last step, step 5, is the most difficult one. If it can be carried out completely, we obtain 'all' solutions, both invariant and non-invariant ones. In general, such a situation is too good to be true. In particular, for the heavenly equation we were not able to solve the system (3.14), (4.13) in general. Instead, we made various simplifying assumptions. The most obvious ones, such as $Y = \bar{Y}$ or $F = 0$, lead to invariant solutions. These we already know, or can obtain by much simpler standard methods. The assumption, or restriction, that leads to non-invariant solutions was $[Y, \bar{Y}] = 0$. The solutions obtained are (7.18) and (7.19), for $\kappa = 1$ and $\kappa = -1$, respectively. Each solution involves two arbitrary holomorphic functions. Only one of them is fundamental. The other is induced by a conformal transformation and can be transformed away (we set $c(z) = z$). In section 8 we show that the solutions are, in general, not invariant under any subgroup of the symmetry group. They reduce to invariant ones only for very special choices of the function $b(z)$, specified in theorems 2 and 3.

It would be interesting to relate the concepts of this paper to that of integrability for non-linear partial differential equations. 'Integrability' means that the considered equation is viewed as an integrability condition for a Lax pair, a pair of linear operators [22, 23]. Here we can view the equations (3.12) as a set of relations between a triplet of linear operators, subject to a non-linear constraint (4.13).

Acknowledgments

References

[1] Lie S 1884 Über Differentialinvarianten *Math. Ann.* **24** 52
[2] Vessiot E 1904 Sur l'integration des sistem differentiels qui admittent des groupes continus de transformations *Acta Math.* **28** 307

[3] Ovsiannikov L V 1982 *Group Analysis of Differential Equations* (New York: Academic)

[4] Martina L 2000 Lie point symmetries of discrete and $SU(\infty)$ Toda field theory *SIDE III: Symmetries and Integrability of Difference Equations vol 25 (CRM Proceedings and Lecture Notes vol 25)* ed D Levi and O Ragnisco (Providence, RI: American Mathematical Society) pp 295–301

[5] Bakas I 1990 Area-preserving diffeomorphisms and higher spin fields in 2 + 1 dimensions *Supermembranes and Physics in* 2 + 1 *Dimensions* ed M Duff, C Pope and E Sezgin (Singapore: World Scientific) pp 352–62

[6] Eguchi T, Gilkey P B and Hanson A J 1980 Gravitation, gauge theory and differential geometry *Phys. Rep.* **66** 213

[7] Ward R S 1990 Einstein–Weil spaces and $SU(\infty)$ Toda fields *Class. Quantum Grav.* **7** L95

[8] Boyer C P and Finley J D III 1982 Killing vectors in self-dual Euclidean Einstein spaces *J. Math. Phys.* **23** 1126–30

[9] Saveliev M V 1989 Integro-differential non-linear equations and continual Lie algebras *Commun. Math. Phys.* **121** 283

[10] Saveliev M V 1992 On the integrability problem of a continuous Toda system *Theor. Math. Phys.* **92** 1024

[11] Nutku Y and Sheftel M B 2001 Differential invariants and group foliation for the complex Monge–Ampère equation *J. Phys. A: Math. Gen.* **34** 137

[12] Alfinito E, Soliani G and Solombrino L 1997 The symmetry structure of the heavenly equation *Lett. Math. Phys.* **41** 379

[13] Boyer C P and Winternitz P 1989 Symmetries of the self-dual Einstein equations: I. The infinite dimensional symmetry group and its low-dimensional subgroups *J. Math. Phys.* **30** 1081

[14] Olver P 1986 *Applications of Lie Groups to Differential Equations* (New York: Springer)

[15] Tresse A 1894 Sur les invariants differentiels des groupes continus de transformations *Acta Math.* **18** 1

[16] Calderbank D M J and Tod P 2001 Einstein metrics, hypercomplex structures and the Toda field equation *Diff. Geom. Appl.* **14** 199–208

[17] Wintenitz P 1993 Group theory and exact solutions of nonlinear partial differential equations *Integrable Systems, Quantum Groups and Quantum Field Theories* (Dordrecht: Kluwer) pp 429–45

[18] Bluman G W and Cole J D 1969 The general similarity solution of the heat equation *J. Math. Mech.* **18** 1025

[19] Clarkson P A and Kruskal M D 1989 New similarity solutions of the Boussinesq equation *J. Math. Phys.* **30** 2201

[20] Levi D and Winternitz P 1989 Non-classical symmetry reduction: example of the Boussinesq equation *J. Phys. A: Math. Gen.* **22** 2915

[21] Clarkson P A and Winternitz P 1999 Symmetry reduction and exact solutions of nonlinear partial differential equations *The Painleve Property, One Century Later* (New York: Springer) pp 597–668

[22] Lax P D 1968 Integrals of nonlinear equations of evolution and solitary waves *Commun. Pure Appl. Math.* **21** 467

[23] Ablowitz M J and Clarkson P A 1991 *Solitons, Nonlinear Evolution Equations and Inverse Scattering* (Cambridge: Cambridge University Press)